

# Lifshitz Tails of Scale-Invariant Theories with Electric Impurities

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We study scale-invariant systems in the presence of Gaussian quenched electric disorder, focusing on the tails of the energy spectra induced by disorder. For relevant disorder we derive asymptotic expressions for the densities of unit-charged states in the tails, positing the existence of saddle points in appropriate disorder integrals. The resultant scalings are dictated by spatial dimensions and dynamical exponents of the systems.

The study of Lifshitz tails – the tails of energy spectra induced by quenched impurities – is an ancient subject [1]. For noninteracting particles obeying the Schrödinger equation with a random potential, there exist various systematic methods for obtaining an asymptotic expression for the density of states deep in the tail [2–8]. The goal of the present paper is to broaden our perspective on Lifshitz tails by exploring a class of systems whose low-energy excitations are governed by scale-invariant theories.

The organization of the paper is as follows. We first set up a scale-invariant theory with quenched electric disorder. After defining the disorder-averaged density of unit-charged states, we argue that there exists a family of saddle points in the appropriate disorder integral. The asymptotic expression for the density of unit-charged states is then derived for large negative energy. We conclude with assertions to be ideally proven in order to rigorously establish the existence of the saddle points postulated herein.

A clean scale-invariant system possesses a dilatation operator  $\hat{D}$  along with a time-translation operator  $\hat{H}_0$  and space-translation operators  $\hat{P}_i$  for  $i = 1, \dots, d$ . These operators obey

$$[\hat{H}_0, \hat{P}_i] = 0, [\hat{P}_i, \hat{P}_j] = 0, [\hat{D}, \hat{P}_i] = i\hat{P}_i, \quad (1)$$

and

$$[\hat{D}, \hat{H}_0] = iz\hat{H}_0 \quad (2)$$

where  $z$  is a dynamical exponent. We suppose that the system has a conserved current with a local “charge” density operator  $\hat{J}^t(\mathbf{x}) = e^{-i\hat{\mathbf{P}} \cdot \mathbf{x}} \hat{J}^t(\mathbf{0}) e^{+i\hat{\mathbf{P}} \cdot \mathbf{x}}$  obeying  $[\hat{J}^t(\mathbf{x}), \hat{J}^t(\mathbf{y})] = 0$  and

$$[\hat{D}, \hat{J}^t(\mathbf{0})] = id\hat{J}^t(\mathbf{0}). \quad (3)$$

A charge number operator  $\hat{Q} \equiv \int d\mathbf{x} \hat{J}^t(\mathbf{x})$  in particular satisfies  $[\hat{H}_0, \hat{Q}] = [\hat{D}, \hat{Q}] = 0$ . We also suppose that there is a local operator  $\hat{\mathcal{O}}_{\text{pro}}^\dagger(\mathbf{x}) = e^{-i\hat{\mathbf{P}} \cdot \mathbf{x}} \hat{\mathcal{O}}_{\text{pro}}^\dagger(\mathbf{0}) e^{+i\hat{\mathbf{P}} \cdot \mathbf{x}}$  with scaling dimension  $\Delta_{\text{pro}}$  and unit, minimal, charge number  $q_{\text{unit}}$ . In other words,

$$[\hat{D}, \hat{\mathcal{O}}_{\text{pro}}^\dagger(\mathbf{0})] = i\Delta_{\text{pro}} \hat{\mathcal{O}}_{\text{pro}}^\dagger(\mathbf{0}) \quad (4)$$

and

$$[\hat{Q}, \hat{\mathcal{O}}_{\text{pro}}^\dagger(\mathbf{0})] = q_{\text{unit}} \hat{\mathcal{O}}_{\text{pro}}^\dagger(\mathbf{0}). \quad (5)$$

We set  $\hbar = 1$  and  $q_{\text{unit}} \equiv 1$  henceforth.

Let us now sprinkle “electric” impurities into the clean system, deforming the Hamiltonian to

$$\hat{H}_{\text{V-random}} = \hat{H}_0 + \int d\mathbf{x} V_{\text{random}}(\mathbf{x}) \hat{J}^t(\mathbf{x}) \quad (6)$$

where for concreteness we suppose that a random potential  $V_{\text{random}}(\mathbf{x})$  obeys Gaussian statistics. An intensive observable  $O$ , when scanned over a macroscopic sample, typically self-averages and we can legitimately estimate it by means of a disorder integral as [9]

$$[O]_{\text{d.a.}} = \int \frac{[DV]}{\mathcal{N}_\sharp} e^{-\frac{1}{2\gamma} \int d\mathbf{x} V^2(\mathbf{x})} O^V. \quad (7)$$

Here,  $O^V$  is the value of the observable in the system governed by  $\hat{H}_V \equiv \hat{H}_0 + \int d\mathbf{x} V(\mathbf{x}) \hat{J}^t(\mathbf{x})$  with a square-integrable potential  $V(\mathbf{x})$ ,  $\mathcal{N}_\sharp \equiv \int [DV] e^{-\frac{1}{2\gamma} \int d\mathbf{x} V^2(\mathbf{x})}$  is the normalization constant for the disorder integral, and  $\gamma$  characterizes the strength of the disorder. For each realization of  $V(\mathbf{x})$  we label eigenstates as

$$\hat{Q}|Q; n\rangle_V = Q|Q; n\rangle_V \quad (8)$$

and

$$\hat{H}_V|Q; n\rangle_V = E_{Q;n}^V|Q; n\rangle_V. \quad (9)$$

We probe the dirty system by injecting a unit-charged excitation through  $\hat{\mathcal{O}}_{\text{pro}}^\dagger$  and observing how it propagates. Specifically we look at a local density of unit-charged states [11] defined via

$$\rho_{\hat{\mathcal{O}}_{\text{pro}}^\dagger}^V(E, \mathbf{x}) \equiv -\frac{1}{\pi} \text{Im} \left\{ G_{\hat{\mathcal{O}}_{\text{pro}}^\dagger}^V(\mathbf{x}, \mathbf{x}; E) \right\} \quad (10)$$

to be disorder-averaged where

$$G_{\hat{\mathcal{O}}_{\text{pro}}^\dagger}^V(\mathbf{x}, \mathbf{y}; E) \equiv -i \int dt e^{iEt} \theta(t) \times {}_V \langle 0; 0 | \hat{\mathcal{O}}_{\text{pro}}^V(t, \mathbf{x}) \hat{\mathcal{O}}_{\text{pro}}^{V\dagger}(0, \mathbf{y}) | 0; 0 \rangle_V \quad (11)$$

with  $\hat{\mathcal{O}}_{\text{pro}}^{V\dagger}(t, \mathbf{x}) \equiv e^{+i\hat{H}_V t} \hat{\mathcal{O}}_{\text{pro}}^\dagger(\mathbf{x}) e^{-i\hat{H}_V t}$ . Here,  $|0; 0\rangle_V$  denotes a state of the lowest energy among states with zero total charge for a given  $V(\mathbf{x})$  [12]. When applied to noninteracting systems, this definition reproduces the standard density of states for a particle excited by  $\hat{\mathcal{O}}_{\text{pro}}^\dagger$ .

The density of unit-charged states  $\rho_{\hat{\mathcal{O}}_{\text{pro}}^{\dagger}}^V(E, \mathbf{x})$  defined above has the spectral representation [13]

$$\sum_n |{}_V\langle Q=1; n | \hat{\mathcal{O}}_{\text{pro}}^{\dagger}(\mathbf{x}) | 0; 0 \rangle_V|^2 \delta(E - E_{1;n}^V + E_{0;0}^V). \quad (12)$$

Contributions for negative energy  $E$ , if any, come from bound states with  $E_{1;n}^V - E_{0;0}^V = E < 0$  and a nonzero overlap  ${}_V\langle Q=1; n | \hat{\mathcal{O}}_{\text{pro}}^{\dagger}(\mathbf{x}) | 0; 0 \rangle_V \neq 0$ . When disorder-averaged, they give rise to a smooth Lifshitz tail. We are interested in the asymptotic behavior of  $[\rho_{\hat{\mathcal{O}}_{\text{pro}}^{\dagger}}(E)]_{\text{d.a.}}$  in the limit of large negative energy  $E$ .

At this point we make two postulates, both of which can be rigorously established for a noninteracting scale-invariant theory with  $z = 2$  [8]. First we assume that for any square-integrable potential  $V(\mathbf{x}) \neq 0$ , when  $d < 2z$ , there exists a state (or states) of lowest energy  $E_{1;0}^V$  among states with a unit charge excited by  $\hat{\mathcal{O}}_{\text{pro}}^{\dagger}$ . Then, as emphasized in [8], the game is to seek a localizing potential which minimizes the cost  $\int d\mathbf{x} V^2(\mathbf{x})$  while still holding a bound state with  $E_{1;0}^V - E_{0;0}^V = E$  for a fixed negative energy  $E$  so that it contributes to  $[\rho_{\hat{\mathcal{O}}_{\text{pro}}^{\dagger}}(E)]_{\text{d.a.}}$ . And here comes our second postulate: for  $d < 2z$ , there exists a family of square-integrable potentials  $V_{\text{saddle}}^E(\mathbf{x})$  which minimizes the cost among all the square-integrable potentials with  $E_{1;0}^V - E_{0;0}^V = E$ . Generically we expect that the competition between the cost, preferring narrower and shallower potential wells, and the demand for trapping a bound state with a fixed negative energy settles into such minimizers. In the saddle-point approximation

$$[\rho_{\hat{\mathcal{O}}_{\text{pro}}^{\dagger}}(E)]_{\text{d.a.}} \sim \exp\left[-\frac{1}{2\gamma} \int d\mathbf{x} \{V_{\text{saddle}}^E(\mathbf{x})\}^2\right] \quad (13)$$

then yields the leading exponential factor.

Armed with the two postulates, we can now obtain the asymptotic expression for the density of unit-charged states in the tail via simple dimensional analysis. Let us be as pedantic as possible, however. First we can use commutation relations to show that

$$e^{-i\lambda\hat{D}} \hat{H}_V e^{i\lambda\hat{D}} = e^{z\lambda} \hat{H}_{V^{(\lambda)}} \quad (14)$$

with

$$V^{(\lambda)}(\mathbf{x}) = e^{-z\lambda} V(e^{-\lambda}\mathbf{x}), \quad (15)$$

from which we deduce that

$$e^{-i\lambda\hat{D}} |Q; n\rangle_V = |Q; n\rangle_{V^{(\lambda)}} \quad (16)$$

with the scaling relation of the spectra

$$E_{Q;n}^{V^{(\lambda)}} = e^{-z\lambda} E_{Q;n}^V. \quad (17)$$

Combined with the scaling relation of the cost

$$\int d\mathbf{x} \{V^{(\lambda)}(\mathbf{x})\}^2 = e^{(d-2z)\lambda} \left[ \int d\mathbf{x} \{V(\mathbf{x})\}^2 \right], \quad (18)$$

we conclude that for  $E^{(\lambda)} = e^{-z\lambda} E$

$$V_{\text{saddle}}^{E^{(\lambda)}}(\mathbf{x}) = \{V_{\text{saddle}}^E(\mathbf{x})\}^{(\lambda)} = e^{-z\lambda} V_{\text{saddle}}^E(e^{-\lambda}\mathbf{x}). \quad (19)$$

From Eqs. (18) and (19) it then follows that

$$\frac{1}{2\gamma} \int d\mathbf{x} \{V_{\text{saddle}}^E(\mathbf{x})\}^2 = \frac{a_0}{g(E)} \quad (20)$$

with the dimensionless constant  $a_0$  and the dimensionless disorder coupling

$$g(E) = \gamma(-E)^{\frac{d}{z}-2}. \quad (21)$$

Thus in the saddle-point approximation

$$[\rho_{\hat{\mathcal{O}}_{\text{pro}}^{\dagger}}(E)]_{\text{d.a.}} \sim e^{-\frac{a_0}{g(E)}} \quad (22)$$

for  $d < 2z$ . This expression is valid in the regime  $E \ll -\gamma^{\frac{z}{2z-d}}$  where the disorder coupling  $g(E)$  is small, akin to the dilute instanton gas limit. We see that the asymptotic scaling of the Lifshitz tail is dictated by the spatial dimension and the dynamical exponent, ordaining the dispersion relation of the low-energy excitations. The scaling dimension  $\Delta_{\text{pro}}$  enters only into the subleading prefactor.

Our result conforms with the existing result [2–8] for a noninteracting scale-invariant system with  $z = 2$ . It is also in accord with the Harris criterion [14] which stipulates that the disorder is relevant for  $d < 2z$ . We can apply our formula to any scale-invariant systems, interacting or not, such as quantum critical materials [15] and relativistic systems with massless charged excitations.

We end with two assertions for unitary scale-invariant theories which, if proven, would ensure the validity of Eq.(22). We state them more strongly here than we did in the body of the paper, respecting close parallels with statements proven for noninteracting particles obeying the Schrödinger equation (for which the second statement becomes equivalent to the instanton problem extensively analyzed in [16–18]).

1. For  $d \leq 2z$ , for any square-integrable potential  $V(\mathbf{x})$ ,  $\hat{H}_V$  admits a normalizable state (or states) of lowest energy for  $Q = 0$  and for  $Q = q_{\text{unit}} \equiv 1$  [19].
2. For  $d < 2z$ , for a fixed negative energy  $E$ , there exists a family of monotone spherically symmetric potentials vanishing at infinity, labeled by the translational collective coordinates, which minimizes the cost  $\int d\mathbf{x} V^2(\mathbf{x})$  among all the square-integrable potentials with  $E_{1;0}^V - E_{0;0}^V = E$ .

On top of proving these statements, it would be valuable to see how far one can generalize the result presented herein: in reality there are quenched disorders other than electric impurities, disorder distributions need not be Gaussian, and we can probe dirty systems through operators with nonminimal charges [20].

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## APPENDIX

In this appendix we derive coupled equations which determine saddle points of the disorder integral. Recall that we are seeking for minima of the cost  $\int d\mathbf{x} V^2(\mathbf{x})$  with the constraint  $E_{1;0}^V - E_{0;0}^V = E$ . Through the introduction of a Lagrange multiplier  $\lambda_0$ , the problem becomes equivalent to the minimization of

$$I[V(\mathbf{x}), \lambda_0] \equiv +\frac{1}{2} \int d\mathbf{x} V^2(\mathbf{x}) + \lambda_0 (E_{1;0}^V - E_{0;0}^V - E) \quad (23)$$

Extremizing it with respect to  $\lambda_0$  reproduces the constraint

$$E_{1;0}^V - E_{0;0}^V = E \quad (24)$$

while extremizing it with respect to  $V(\mathbf{x})$  yields

$$V(\mathbf{x}) = -\lambda_0 \left[ {}_V\langle 1;0 | \hat{J}^t(\mathbf{x}) | 1;0 \rangle_V - {}_V\langle 0;0 | \hat{J}^t(\mathbf{x}) | 0;0 \rangle_V \right]. \quad (25)$$

Here,

$$\hat{H}_V |0;0\rangle_V = E_{0;0}^V |0;0\rangle_V \quad (26)$$

and

$$\hat{H}_V |1;0\rangle_V = E_{1;0}^V |1;0\rangle_V, \quad (27)$$

and we used the Hellmann-Feynman relation [21, 22]

$$\frac{\delta}{\delta V(\mathbf{x})} E_{Q;0}^V = {}_V\langle Q;0 | \hat{J}^t(\mathbf{x}) | Q;0 \rangle_V. \quad (28)$$

Solving the coupled equations (24), (25), (26), and (27), we obtain saddle points of the disorder integral in question. We then further seek for a family of saddle points which minimizes the cost among all the saddles.

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